Journal of Sound and Vibration (2001) **242**(5), 855–866
doi:10.1006/jsvi.2000.3386, available online at http://www.idealibrary.com on IDEAL®

MULTIPLE RESONANT OR NON-RESONANT PARAMETRIC EXCITATIONS FOR NONLINEAR OSCILLATORS

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(*Received* 22 *March* 2000, *and in final form* 7 *September* 2000)

The transient and steady state response of a very general non-linear oscillator subject to a finite number of parametric excitations is considered by the asymptotic perturbation method. Three main cases are examined: (1) the parametric excitations frequencies are not close to each other or close to the principal parametric resonance of the oscillator; (2) the parametric excitations frequencies are close to each other but not close to the principal resonance; and (3) all the parametric excitations frequencies are close to the principal resonance. Both the conditions for the quenching of the oscillation and the conditions for its persistence are determined. The main conclusion is that the oscillation in systems with one degree of freedom cannot be fully quenched due to the action of parametric excitation, because the only change is a shift in the oscillator frequencies. Analytical approximate results are checked by numerical integration.

1. INTRODUCTION

The transient and steady state response of a non-linear oscillator subject to a finite number of parametric excitations is considered. The relevant non-linear ordinary differential equation is

$$\ddot{X}(t) + (1 + F(t))X(t) + a\dot{X}(t) + bX^2(t) + cX(t)\dot{X}(t)$$

$$+ d\dot{X}^2(t) + eX^3(t) + f\dot{X}(t)X^2(t) + g\dot{X}^2(t)X(t) + h\dot{X}^3(t) = 0, \tag{1}$$

where the dots denote differentiation with respect to the non-dimensional time and $F(t)$ is a finite sum of N harmonic parametric excitations of the form

$$F(t) = \sum_{m=1}^{N} 2A_m \cos(\Omega_m t), \tag{2}$$

where A_m is the amplitude and Ω_m is the non-dimensional frequency of the mth component of $F(t)$. (All times are referred to the time scale $1/\omega$, where ω is the natural frequency of the linearized oscillator.) The general non-linear oscillator (1) can be self-excited for some values of the linear and non-linear terms and its asymptotic behavior will be modified by the interaction between the multi-frequency parametric excitation and the self-excitation.

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Only the case $N > 1$ is considered and attention is especially devoted to the modifications induced by the non-linear terms on the solution of the linearized and undamped version of equation (1):

$$\ddot{X}(t) + \left(1 + \sum_{m=1}^{N} 2A_m \cos(\Omega_m t)\right) X(t) = 0, \tag{3}$$

which is the well-known Mathieu equation [1–3]. For small values of the excitation amplitudes A_m, the solution of equation (3) can be written as a perturbative expansion

$$X(t) = 2\rho_0 \cos(-t + \vartheta_0) + \sum_{m=1}^{N} \frac{2A_m \rho_0}{(\Omega_m^2 + 2\Omega_m)} \cos((\Omega_m + 1)t - \vartheta_0)$$

$$+ \sum_{m=1}^{N} \frac{2A_m \rho_0}{(\Omega_m^2 - 2\Omega_m)} \cos((\Omega_m - 1)t + \vartheta_0) + O(A_m^2), \tag{4}$$

where ρ_0, ϑ_0 are fixed by the initial conditions and

$$\Omega_m = A_m^2/(4 - \Omega_m^2). \tag{5}$$

This solution is the sum of the free oscillation and the forced oscillation. It is essential to discover if both the free oscillation, i.e., the first term of the right-hand side (r.h.s.) of equation (4), and the forced oscillation will persist or decay ("quenching"), when the non-linear terms are active.

Equation (1) contains as particular cases well-known oscillators: the van der Pol oscillator ($a < 0, f > 0$ and all the other parameters zero), the Duffing oscillator ($a, e \neq 0$ and all the other parameters zero) and so on.

In particular, the van der Pol oscillator and many other special cases have been studied extensively for $N = 1$ and $N = 2$ [1–7]. Atallah and Geer used the method of multiple time scales to study the van der Pol and the Duffing oscillator with external excitations and $N > 2$ [8], but with no comparison with numerical results. Maccari [9] extended that particular study to a more general oscillator and compared analytical and numerical results. The most important finding was that if the external excitation frequencies are not close to the primary resonance frequency, the amplitude of the free oscillation will decay exponentially in time, if the amplitude of the forcing term is sufficiently large, but will otherwise approach a constant value.

The paper is organized as follows. In section 2, by using the asymptotic perturbation method, the non-linear oscillator (1) is studied when the parametric excitations are not close to each other and not close to the principal parametric resonance. In section 3 approximate solutions are derived when the parametric excitations are close to each other, but not close to the principal parametric resonance. Finally, in section 4 parametric excitations near the principal resonance are considered.

The general approach is inspired by the asymptotic perturbation method [10–12] for discrete dynamical systems and the formal perturbation solution is carried out to the lowest order approximation. To the best of the author's knowledge, the parametrically excited non-linear oscillator (1) has not been examined with multiple scales or harmonic balance methods.

The results of this study can be judiciously applied to determine the transient and steady state response of a non-linear oscillator subject to a virtually arbitrary signal, because it is well known that any signal over a specific time period can be approximated by a finite number of trigonometric terms.

The paper closes with some conclusions and directions for future work, in section 5.

2. PARAMETRIC EXCITATIONS NOT CLOSE TO EACH OTHER

In this section the parametric excitations are supposed to be not close to each other or close to the principal resonance. Some approximate solutions are constructed by using the asymptotic perturbation method. This method comes from a similar method employed in non-linear partial differential equations and is based on the detailed computation of the interaction of harmonic solutions of the linear part of the differential equation, because of the presence of the non-linear terms.

By means of the temporal rescaling

$$\tau = \varepsilon^2 t, \tag{6}$$

attention is devoted to the asymptotic behavior of the solution: when $t \to \infty$ and $\varepsilon \to 0$, then τ can assume finite values.

The required solution of equation (1) can be expressed as a perturbation expansion, based on the parameter ε (a bookkeeping device that can be set equal to zero in the final analysis):

$$\begin{aligned} X(t) = [&\varepsilon\psi_{10}(\tau;\varepsilon)\exp(-\mathrm{i}t) + \varepsilon^2(\tfrac{1}{2}\psi_{00}(\tau;\varepsilon) + \psi_{20}(\tau;\varepsilon)\exp(-2\mathrm{i}t)) \\ &+ \varepsilon^3(\psi_{30}(\tau;\varepsilon)\exp(-3\mathrm{i}t) + \sum_{m=1}^{N}\psi_{1m}(\tau;\varepsilon)\exp(-\mathrm{i}t - \mathrm{i}\Omega_m t) \\ &+ \sum_{m=1}^{N}\psi_{1-m}(\tau;\varepsilon)\exp(-\mathrm{i}t + \mathrm{i}\Omega_m t) + \mathrm{c.c.}] + O(\varepsilon^4). \end{aligned} \tag{7}$$

Here c.c. stands for complex conjugate, $\psi_{np}(\tau,\varepsilon) = \psi^*_{-n-p}(\tau,\varepsilon)$, $(n = 1, 2;\ p = 0, 1, \ldots, N)$, because $X(t)$ is real (the asterisk denotes complex conjugate).

The function $\psi_{np}(\tau,\varepsilon)$ depends on the parameter ε and it is supposed that the limit of the ψ_{np}'s for $\varepsilon \to 0$ exists and is finite. The expansion (7) can be substituted into the differential equation (1) so as to obtain separate equations for each couple n, p and subsequently, coefficients of like powers of ε are equated.

By using the asymptotic perturbation method the advantages of the harmonic balance method (see equation (7)) and the multiple scales technique (see equation (6)) are simultaneously taken into account. The method is constructive in a local sense, i.e., near an equilibrium point of the oscillator, so that one can reconstruct the general motion of the system.

With $\psi(\tau)$, $\varphi_m(\tau)$ indicating the limits of $\psi_{10}(\tau,\varepsilon)$, $\psi_{1m}(\tau;\varepsilon)$ when $\varepsilon \to 0$, the following equation is obtained for $n = 1$, $p = 0$:

$$\begin{aligned} &2\mathrm{i}\psi_\tau\varepsilon^3 + \mathrm{i}a\psi\varepsilon^3 - (2b - \mathrm{i}c)(\psi_0\psi\varepsilon^{1+r} + \psi_2\psi^*\varepsilon^3) - 4\mathrm{d}\psi_2\psi^*\varepsilon^3 \\ &\quad - (3e - \mathrm{i}f - 3\mathrm{i}h + g)|\psi|^2\psi\varepsilon^3 - \sum_{m=1}^{N} A_m(\varphi_m + \tilde{\varphi}_m) = 0. \end{aligned} \tag{8}$$

For $n = 1$, $p \neq 0$, one obtained

$$(1 + \Omega_m)^2 \varphi_m - \varphi_m - A_m \psi = 0, \tag{9}$$

$$(1 - \Omega_m)^2 \tilde{\varphi}_m - \tilde{\varphi}_m - A_m \psi = 0. \tag{10}$$

For $n = 0$, $p = 0$ and $n = 2$, $p = 0$, equation (1) yields

$$\psi_0 = -2(b + d)|\psi|^2, \qquad \psi_2 = \frac{(b - d - \mathrm{i}c)}{3}\psi^2, \tag{11, 12}$$

while equations (9) and (10) yield

$$\varphi_m = \frac{A_m}{\Omega_m^2 + 2\Omega_m}\psi, \qquad \varphi_{-m} = \frac{A_m}{\Omega_m^2 - 2\Omega_m}\psi. \tag{13}$$

Equations (11)–(13) can be substituted into equation (6) and thus the following model equation is obtained:

$$\psi_\tau = (\alpha_1 + \mathrm{i}\alpha_2)\psi + (\beta_1 + \mathrm{i}\beta_2)|\psi|^2\psi. \tag{14}$$

Here

$$\alpha_1 = -\frac{a}{2}, \qquad \alpha_2 = \sum_{m=1}^{N} \frac{A_m^2}{4 - \Omega_m^2}, \tag{15}$$

$$\beta_1 = \frac{1}{2}\left(bc + \frac{cd}{3} - f - 3h\right), \tag{16}$$

$$\beta_2 = \frac{1}{2}\left(\frac{10}{3}(b^2 + bd) + \frac{c^2 + 4d^2}{3} - 3e - g\right). \tag{17}$$

By means of the standard substitution

$$\psi(\tau) = \rho(\tau)\exp(\mathrm{i}\vartheta(\tau)), \tag{18}$$

equation (14) can be separated into two parts:

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \alpha_1\rho + \beta_1\rho^3, \qquad \frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \alpha_2 + \beta_2\rho^2. \tag{19, 20}$$

The approximate solution that is good to the order of ε^2 is

$$\begin{aligned} X(t) &= 2\rho(t)\cos(-t + \vartheta(t)) - 2(b + d)\rho^2(t) \\ &\quad + \frac{2}{3}(b - d)\rho^2(t)\cos(-2t + 2\vartheta(t)) + \frac{2}{3}cp^2(t)\sin(-2t + 2\vartheta(t)) \\ &\quad + \sum_{m=1}^{N}\left[\frac{2A_m\rho(t)\cos((\Omega_m + 1)t - \vartheta(t))}{(\Omega_m^2 + 2\Omega_m)} + \frac{2A_m\rho(t)\cos((\Omega_m - 1)t + \vartheta(t))}{(\Omega_m^2 - 2\Omega_m)}\right]. \end{aligned} \tag{21}$$

Note that the temporal evolution of $\rho(\tau)$ does not depend on $\vartheta(\tau)$ and then equation (19) can be easily integrated:

$$\rho(\tau) = \rho_0 \left[\left(1 + \frac{\beta_1 \rho_0^2}{\alpha_1} \right) \exp(-2\alpha_1 \tau) - \frac{\beta_1 \rho_0^2}{\alpha_1} \right]^{1/2}. \tag{22}$$

From inspection of equation (22), it is easily deduced that $\rho(\tau)$ diverges when

$$\tau = \tau_0 = \left(\frac{1}{2\alpha_1} \right) \log \left[\frac{\beta_1 \rho_0^2 + \alpha_1}{\beta_1 \rho_0^2} \right] \tag{23}$$

if $\beta_1 > 0$, $\alpha_1 + \beta_1 \rho_0^2 > 0$. Four cases can be distinguished:

(1) $\alpha_1 > 0$, $\beta_1 > 0$. Stable equilibrium points do not exist and then the solution diverges (obviously the approximation is not valid for $\tau \cong \tau_0$).
(2) $\alpha_1 < 0$, $\beta_1 < 0$. The origin is an asymptotically stable equilibrium point and $\rho(\tau)$ approaches zero as τ goes to infinity ("quenching" of the oscillation).
(3) $\alpha_1 > 0$, $\beta_1 < 0$. $\rho(\tau)$ approaches the stable equilibrium point

$$\rho_1 = (-\alpha_1/\beta_1)^{1/2} \tag{24}$$

and then the oscillation is always present. Unless the Ω_i are all rational numbers, i.e., commensurable with ω, the motion will be quasi-periodic; the asymptotic solution is

$$\begin{aligned} X(t) = {} & 2\rho_1 \cos((1-\tilde{\omega})t) - 2(b+d)\rho_1^2 \\ & + \frac{2}{3}(b-d)\rho_1^2 \cos(2(1-\tilde{\omega})t) + \frac{2}{3} c\rho_1^2 \sin(2(1-\tilde{\omega})t) \\ & + \sum_{m=1}^{N} \left[\frac{2A_m \rho_1 \cos((\Omega_m + 1 - \tilde{\omega})t)}{(\Omega_m^2 + 2\Omega_m)} + \frac{2A_m \rho_1 \cos((\Omega_m - 1 + \tilde{\omega})t)}{(\Omega_m^2 - 2\Omega_m)} \right], \end{aligned} \tag{25}$$

where

$$\tilde{\omega} = \alpha_2 + \beta_2 \rho_1^2. \tag{26}$$

(4) $\alpha_1 < 0$, $\beta_1 > 0$. The origin is a stable equilibrium point ("quenching" of the oscillation) and the solution (24) exists but it is unstable. If $\rho_0 > \rho_1$ then the solution diverges when $\tau \cong \tau_0$.

The main conclusion of the above discussion is that the oscillation in systems governed by equation (1) cannot be fully quenched by the parametric excitation (the coefficients of the various parametric excitations are present only in the term α_2 (see equation (15)). However, the multi-frequency parametric excitation can modify the oscillator frequencies, as in equations (25) and (26).

Numerical integration of equation (1) can be used to check the qualitative picture which emerges from the preceding analysis. For example, the numerical solution compared with the approximate solution (25) for case (3) is shown in Figure 1. The mean difference between the two solutions is (0·0032), i.e., of order ε^3 as expected.

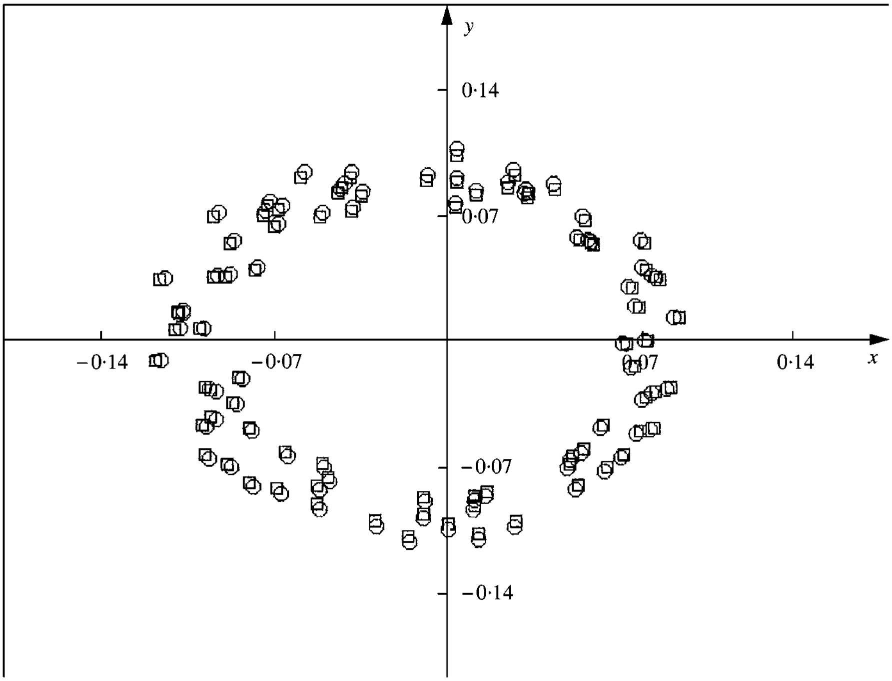

Figure 1. Comparison between numerical (rectangles) and analytical (circles) solutions in the $(X, \dot{X} = Y)$ plane. Values of parameters: $a = -0{\cdot}01$, $b = 1{\cdot}5$, $c = -1{\cdot}0$, $d = 1{\cdot}0$, $e = 0{\cdot}5$, $f = 0{\cdot}6$, $g = -0{\cdot}5$, $h = 1{\cdot}0$. Parametric excitation frequencies not close to each other and not close to the principal resonance: $\Omega_1 = \sqrt{3}$, $\Omega_2 = \sqrt{5}$, $\Omega_3 = \sqrt{7}$. Amplitudes of the external excitations: $A_1 = 0{\cdot}05$, $A_2 = 0{\cdot}03$, $A_3 = 0{\cdot}02$.

A physical example of the previous analysis can be furnished by the van der Pol oscillator ($a < 0, f > 0$ and all other parameters zero). In this case, equations (15)–(17) yield

$$\alpha_1 = -\frac{a}{2}, \qquad \alpha_2 = \sum_{m=1}^{N} \frac{A_m^2}{4 - \Omega_m^2}, \tag{27}$$

$$\beta_1 = -\frac{f}{2}, \qquad \beta_2 = 0. \tag{28}$$

The van der Pol oscillator belongs to case (3): the oscillation is always present with amplitude

$$\rho_1 = (-a/f)^{1/2}, \tag{29}$$

and the approximate asymptotic solution is

$$X(t) = 2\rho_1 \cos((1 - \tilde{\omega})\,t) + \sum_{m=1}^{N} \left[\frac{2A_m \rho_1 \cos((\Omega_m + 1 - \tilde{\omega})t)}{(\Omega_m^2 + 2\Omega_m)} + \frac{2A_m \rho_1 \cos((1 - \Omega_m - \tilde{\omega})t)}{(\Omega_m^2 - 2\Omega_m)} \right], \tag{30}$$

where

$$\tilde{\omega} = \sum_{m=1}^{N} \frac{A_m^2}{4 - \Omega_m^2}. \tag{31}$$

The multi-frequency parametric excitation causes a shift $\tilde{\omega}$ in the oscillator natural frequency 1. This shift is proportional to the square of the amplitudes of the various parametric excitations. On the other hand, the parametric components of the motions are proportional to the amplitude ρ_1 of the fundamental oscillation.

3. THE APPROXIMATE SOLUTION WITH THE FREQUENCIES CLOSE TO EACH OTHER

The results of the previous section can be extended to the case when the parametric excitation frequencies are close to each other, but not close to the principal resonance.

The detunings σ_m can be introduced through the relation

$$\Omega_m = \Omega + \varepsilon^2 \sigma_m, \quad m = 1, \ldots, N, \tag{32}$$

where Ω is a fixed frequency not close to the principal resonance. The detunings σ_m measure the differences of the frequencies from each other. By substituting equation (32) into equation (2), the parametric excitation $F(t)$ can be written as

$$F(t) = \varepsilon \frac{\exp(\mathrm{i}\Omega t)}{(1 - \Omega^2)} \sum_{m=1}^{N} A_m \exp(\mathrm{i}\varepsilon^2 \sigma_m \tau) + \text{c.c.} + \mathrm{O}(\varepsilon^3). \tag{33}$$

The non-linear oscillator is then subject to an applied force with frequency Ω and with an amplitude that is a slowly varying function of time.

The same method as that in section 2 can be applied and equations (19) and (20) are again obtained but now with

$$\alpha_2 = \frac{1}{(1 - \Omega^2)(4 - \Omega^2)} \sum_{m=1}^{N} A_m^2 + \frac{1}{(1 - \Omega^2)(4 - \Omega^2)} \sum_{m,n=1\lfloor m \neq n \rfloor}^{N} A_m A_n \exp(\mathrm{i}(\sigma_m - \sigma_n)t), \tag{34}$$

and α_1, β_1, β_2 unchanged. Also, in this case the evolution of $\rho(\tau)$ does not depend on $\vartheta(\tau)$, but the difference now is that α_2 is explicitly dependent on τ.

A simple integration shows that

$$\vartheta(\tau) = \vartheta_0 + \beta_2 \int_0^{\tau} \rho^2(\tau')\,\mathrm{d}\tau' + \frac{1}{(1 - \Omega^2)(4 - \Omega^2)} \left(\tau \sum_{m=1}^{N} A_m^2 - \mathrm{i} \sum_{m,n=1(m \neq n)}^{N} A_m A_n \frac{(\exp(\mathrm{i}(\sigma_m - \sigma_n)\tau) - 1)}{(\sigma_m - \sigma_n)} \right). \tag{35}$$

The temporal evolution of $\rho(\tau)$ is now exactly as in the previous section and the same conclusions are valid for the quenching of the oscillation.

The behavior of ϑ as τ becomes large is important in case (3) of the previous section and can be easily determined from equation (35). The asymptotic behavior is

$$\vartheta(\tau) = \tilde{\omega}\tau + \delta(\tau), \tag{36}$$

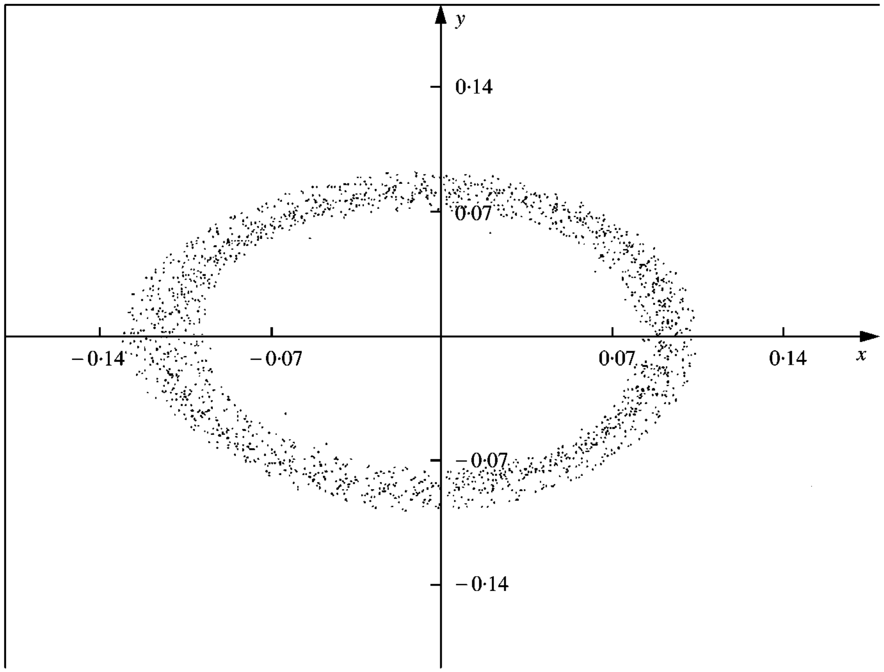

Figure 2. Associated map of the non-linear oscillator (1) with the parametric excitation frequencies close to each other but not close to the principal resonance: $\Omega_1 = \sqrt{3}$, $\Omega_2 = \sqrt{3{\cdot}1}$, $\Omega_3 = \sqrt{3{\cdot}2}$. Values of parameters: $a = -0{\cdot}01$, $b = 1{\cdot}5$, $c = -1{\cdot}0$, $d = 1{\cdot}0$, $e = 0{\cdot}5$, $f = 0{\cdot}6$, $g = -0{\cdot}5$, $h = 1{\cdot}0$. Amplitudes of the external excitations: $A_1 = 0{\cdot}05$, $A_2 = 0{\cdot}03$, $A_3 = 0{\cdot}02$.

where

$$\tilde{\omega} = \beta_2 \rho_1^2 + \frac{1}{(1-\Omega^2)(4-\Omega^2)} \sum_{m=1}^{N} A_m^2 \tag{37}$$

and $\delta(\tau)$ indicates the oscillating part.

The approximate asymptotic solution is given by equation (25) but with $\vartheta(\tau)$ as in equation (36). While in the previous section the asymptotic motion of the solution is quasi-periodic with a finite number of frequencies, now, due to the oscillating part of $\vartheta(\tau)$, an infinite number of frequencies are excited.

The associated map of the non-autonomous equation (1) obtained with the values ($X(0)$, $Y(0)$), ($X(T)$, $Y(T)$), ($X(2T)$, $Y(2T)$), …, where T is the period of the parametric excitation ($T = 2\pi/\Omega$), is shown in Figure 2. The numerical solution has been compared with the approximate solution (but is not shown in the figure). The mean difference between the two solutions is 0·0036, i.e., of order ε^3 as expected.

4. PARAMETRIC EXCITATION FREQUENCIES NEAR PRINCIPAL RESONANCE

If the frequency of each component of the parametric excitation term is near the principal resonant frequency of the oscillator, then

$$\Omega_m = 2 + \varepsilon^2 \sigma_m, \quad m = 1, \ldots, N, \tag{38}$$

where σ_m measures the differences of the frequencies from the principal resonance of the oscillator. Upon using equation (38) in equation (2), the parametric excitation $F(t)$ becomes

$$F(t) = \varepsilon^2 \exp(2\mathrm{i}t) \sum_{m=1}^{N} A_m \exp(\mathrm{i}\varepsilon^2 \sigma_m \tau) + \text{c.c.} \tag{39}$$

The non-linear oscillator is then subject to a parametric excitation with N different frequencies and amplitudes, which are supposed to be of order ε^2, because in this section only the principal resonance zone is considered.

The approximate solution is in the form

$$X(t) = [\varepsilon\psi(\tau;\varepsilon)\exp(-\mathrm{i}t) + \varepsilon^2(\tfrac{1}{2}\psi_0(\tau;\varepsilon) + \psi_2(\tau;\varepsilon)\exp(-2\mathrm{i}t)) + \text{c.c.}] + O(\varepsilon^3), \tag{40}$$

with the same conventions as in equation (7).

The solution (40) is substituted into equation (1) so as to obtain different equations for each n and subsequently coefficients of like powers of ε are equated. For $n = 1$ the result is

$$\begin{aligned} &2\mathrm{i}\psi_\tau + \mathrm{i}a\psi - (2b - \mathrm{i}c)(\psi_0\psi + \psi_2\psi^*) - 4\,\mathrm{d}\psi_2\psi^* \\ &- (3e - \mathrm{i}f - 3\mathrm{i}h + g)|\psi|^2\psi + \psi^* \sum_{m=1}^{N} A_m \exp(-\mathrm{i}\sigma_m\tau) = 0. \end{aligned} \tag{41}$$

The details of the calculation are not given and only the final results are furnished. By means of the substitution (18), the equations for the amplitude and the phase are obtained,

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \alpha_1\rho + \beta_1\rho^3 + \frac{\rho}{2}\sum_{i=1}^{N} A_i \sin(\sigma_i\tau + 2\vartheta), \tag{42}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \beta_2\rho^2 + \frac{1}{2}\sum_{i=1}^{N} A_i \cos(\sigma_i\tau + 2\vartheta), \tag{43}$$

where α_1 and β_1, β_2 are given by equations (15)–(17).

The difference with respect to the preceding cases is that now equations (42) and (43) are two coupled non-linear differential equations, which must be integrated numerically.

However, a very interesting behavior is observed if $\alpha_1 > 0$ and $\beta_1 < 0$ and

$$\beta_2\rho_1^2 \gg \sum_{i=1}^{N} |A_i|, \quad \text{with } \rho_1 = (-\alpha_1/\beta_1)^{1/2}, \tag{44}$$

i.e., for weak parametric excitations. In this case, at least for initial conditions near ρ_1, the system (42) and (43) can be approximated by

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = -2\alpha_1(\rho - \rho_1) + \frac{\rho_1}{2}\sum_{i=1}^{N} A_i \sin(\sigma_i\tau + 2\hat{\Omega}\tau + 2\vartheta_0), \tag{45}$$

$$\vartheta(\tau) = \hat{\Omega}\tau + \vartheta_0, \tag{46}$$

where $\hat{\Omega} = \beta_2 \rho_1^2$. The solution of equation (45) is

$$\rho(t) = \rho_1 + \rho_0 \exp(-2\alpha_1 t) + \frac{\rho_1}{4} \sum_{m=1}^{N}$$

$$\frac{A_m[4\alpha_1 \sin(\tilde{\Omega}_m t + 2\vartheta_0) - 2\tilde{\Omega}_m \cos(\tilde{\Omega}_m t + 2\vartheta_0) + \exp(-2\alpha_1 t)(2\tilde{\Omega}_m \cos(2\vartheta_0) - 4\alpha_1 \sin(2\vartheta_0)]}{4\alpha_1^2 + \tilde{\Omega}_m^2}, \tag{47}$$

where ρ_0 is fixed by the initial conditions and

$$\tilde{\Omega}_m = 2\hat{\Omega} + \sigma_m. \tag{48}$$

The asymptotic behavior of equation (47) is

$$\rho(t) = \rho_1 + \frac{\rho_1}{4} \sum_{m=1}^{N} \frac{A_m[4\alpha \sin(\tilde{\Omega}_m t + 2\vartheta_0) - 2\tilde{\Omega}_m \cos(\tilde{\Omega}_m t + 2\vartheta_0)]}{4\alpha_1^2 + \tilde{\Omega}_m^2} \tag{49}$$

and then the solution amplitude is slightly modulated with N different frequencies depending on the non-linear terms and parametric excitation frequencies.

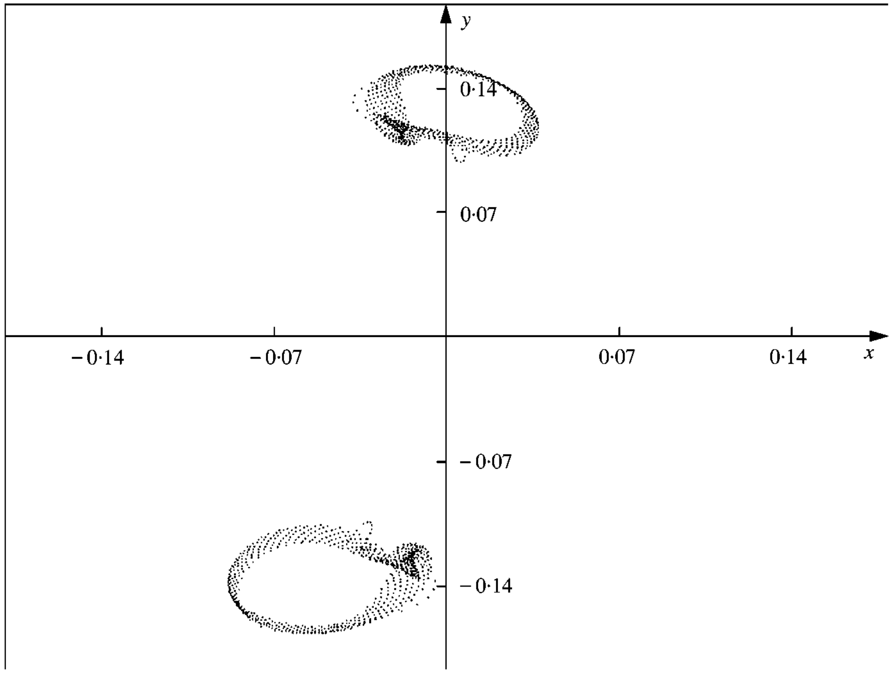

Figure 3. Associated map of the non-linear oscillator (1) with the parametric excitation frequencies close to each other and close to the principal resonance: $\Omega_1 = \sqrt{1{\cdot}3}$, $\Omega_2 = \sqrt{1{\cdot}2}$, $\Omega_3 = \sqrt{1{\cdot}1}$. Values of parameters: $a = -0{\cdot}01$, $b = 1{\cdot}5$, $c = -1{\cdot}0$, $d = 1{\cdot}0$, $e = 0{\cdot}5$, $f = 0{\cdot}6$, $g = -0{\cdot}5$, $h = 1{\cdot}0$. Amplitudes of the external excitations: $A_1 = 0{\cdot}05$, $A_2 = 0{\cdot}03$, $A_3 = 0{\cdot}02$.

The approximate solution that is good to the order of ε^2 is

$$X(t) = 2\rho(t)\cos((1-\tilde{\Omega})t + \vartheta_0) - 2(b+d)\rho^2(t) + \frac{2}{3}(b-d)\rho^2(t)\cos(2(1-\hat{\Omega})t + 2\vartheta_0) + \tfrac{2}{3}cp^2(t)\sin(2(1-\hat{\Omega})t + 2\vartheta_0), \qquad (50)$$

where $\rho(t)$ is given by equation (49).

The associated map of the non-autonomous equation (1) obtained with the values ($X(0)$, $Y(0)$), ($X(T)$, $Y(T)$), ($X(2T)$, $Y(2T)$), …, where T is the period of the parametric excitation ($T = \pi$, $\Omega = 2$), is shown in Figure 3. The numerical solution has been compared with the approximate solution (50) (but is not shown in the figure). The mean difference between the two solutions is 0·0033, i.e., of order ε^3 as expected.

5. CONCLUSIONS

The asymptotic perturbation method has been used to consider the transient and steady state response of a general non-linear oscillator under a finite number of harmonic parametric excitations.

An important feature of this method is that it provides quantitative results regarding dynamic behavior, in contrast to much of the current work in dynamical systems theory, which is concerned with qualitative behavior.

Three different cases are investigated and the corresponding analytical results are compared to numerical simulations. If the parametric excitation frequencies are not close to each other or close to the principal resonance, then the original oscillation can vanish ("quenching") or maintain a finite value, when non-linear terms are added. The multi-frequency parametric excitation is only able to change the oscillator frequencies, because the quenching is determined by some linear and non-linear terms (coefficients a, b, c, d, f, h in equation (1)).

If the parametric excitations frequencies are all close to a particular frequency Ω, the "quenching" is possible but in certain cases the oscillation is quasiperiodic with an infinite number of frequencies determined by the detuning parameters.

When the parametric excitation frequencies are close to the principal resonance frequency, then both the amplitude and the phase of the oscillation can eventually oscillate with a frequency that is determined by both the parametric excitation amplitudes A_i and the detuning parameters σ_i. Approximate analytical solutions can be derived in a particular case, when the parametric excitations are very weak. The solution amplitude is slightly modulated with N different frequencies depending on the non-linear terms and the parametric excitation frequencies.

A possible extension of the present study is given by the calculation of the second order approximation solution. However, in this case the amount and complexity of the algebraic computations required increase in a very dramatic manner and the use of symbolic manipulation systems is strongly recommended.

Another extension can be the study of synchronization effects in two-degree-of-freedom systems with multi-frequency parametric excitation. Previous papers have dealt mostly with periodic solutions (synchronized or mode-locked states) in the presence of a single-frequency parametric excitation.

REFERENCES

1. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations*. New York: Wiley.
2. A. H. NAYFEH and B. BALACHANDRAN 1995 *Applied Nonlinear Dynamics*. New York: Wiley.

3. S. H. Strogatz 1996 *Nonlinear Dynamics and Chaos*. Reading, MA: Addison-Wesley.
4. W. M. Mansour 1972 *Journal of Sound and Vibration* **25**, 395–405. Quenching of limit cycles of a van der Pol oscillator.
5. A Tondl 1975 *Journal of Sound and Vibration* **42,** 251–260. Quenching of self-excited vibrations: equilibrium aspects.
6. A. Tondl 1975 *Journal of Sound and Vibration* **42**, 261–271. Quenching of self-excited vibrations: one- and two-frequency vibrations.
7. Y. Shinohara, A. Kohda and T. Mitsui 1984 *Journal of Mathematics*, *Tokushima University* **18**, 1–9. On quasiperiodic solutions to van der Pol equation.
8. G. C. Atallah and J. Geer 1987 *International Journal of Non-Linear Mechanics* **22**, 439–449. Non-linear oscillations with multiple forcing terms.
9. A. Maccari 1999 *International Journal of Non-Linear Mechanics* **34**, 27–34. Non-linear oscillations with multiple resonant or non-resonant forcing terms.
10. A. Maccari 1996 *II Nuovo Cimento* **111B**, 917–930. The non-local oscillator.
11. A. Maccari 1998 *Nonlinear Dynamics* **15**, 329–343. Approximate solution of a class of nonlinear oscillators in resonance with a periodic excitation.
12. A. Maccari 2000 *International Journal of Non-Linear Mechanics* **35**, 239–262. Modulated motion and infinite-period homoclinic bifurcation for parametrically excited Liénard systems.